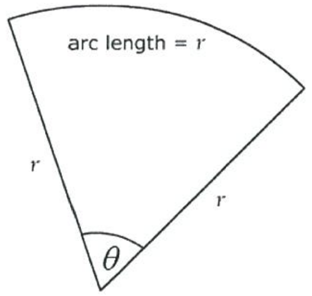
Year 13 Trigonometry Workbook

Answers.

Radians and Degrees



$\theta = 1$ radian

Angles can be measured in angles and radians.

A radian is the angle made by taking the radius and wrapping it along the edge of a circle.

If θ is in radians, then $\theta = \dfrac{\text{arc length}}{\text{radius}}$, (a number without a unit)

Converting between radians and degrees

The circumference of a whole circle is $2\pi r$. For a full circle:

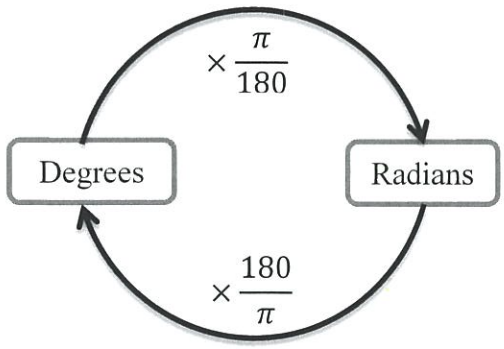
In degrees $\quad \theta = 360^{\circ}$.

In radians $\quad \theta = \dfrac{\text{arc length}}{\text{radius}} = \dfrac{2\pi r}{r} = 2\pi$

So $\quad 360^{\circ} = 2\pi$ radians, or $180^{\circ} = \pi$ radians.

To convert from radians to degrees, multiply by $\dfrac{180}{\pi}$.

To convert from degrees to radians, multiply by $\dfrac{\pi}{180}$.



Example:
Convert 315° to radians.
Leave your answer in terms of π.

Ans $\quad 315 \times \dfrac{\pi}{180} = \dfrac{315\pi}{180} = \dfrac{7\pi}{4}$

Example:
Convert $\frac{2\pi}{3}$ to degrees

Ans $\quad \dfrac{2\pi}{3} \times \dfrac{180}{\pi} = \dfrac{360\pi}{3\pi} = 120^{\circ}$

Some useful conversions between degrees and radians are below, complete the table:

Angle in radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Angle in degrees	30°	45°	60°	90°	180°	270°	360°

Exercise I: Angle Conversions

1. Convert the following angles from degrees to radians, leaving answers as multiples of π

a. 90^{o} $\frac{90}{180} \times \pi = \frac{\pi}{2}$

b. 225^{o} $\frac{225}{180}\pi = \frac{5\pi}{4}$

c. 162^{o} $\times \frac{\pi}{180} = \frac{9\pi}{10}$

d. 15^{o} $\times \frac{\pi}{180} = \frac{\pi}{12}$

2. Convert the following angles from radians to degrees, rounding to 2d.p. where necessary

a. 2.3 rad $\times \frac{180}{\pi} = 131.8^{o}$

b. $\frac{4\pi}{3}$ rad $\times \frac{180}{\pi} = 240^{o}$

c. $\frac{3\pi}{10}$rad $\times \frac{180}{\pi} = 54^{o}$

d. 5.1 rad $= 292.2^{o}$

Graphs of Trigonometric Functions

Definitions

The **period** of a trig graph is the minimum cycle before a graph repeats itself
The **amplitude** of a trig graph is the maximum height either side of the central position
The **frequency** is the number of complete cycles the occur in 2π radians or 360 degrees ($=\frac{2\pi}{period}$)

The three main trig graphs are $y = \sin \boldsymbol{x}$; $y = \cos \boldsymbol{x}$; $y = \tan \boldsymbol{x}$:

Properties of trig graphs

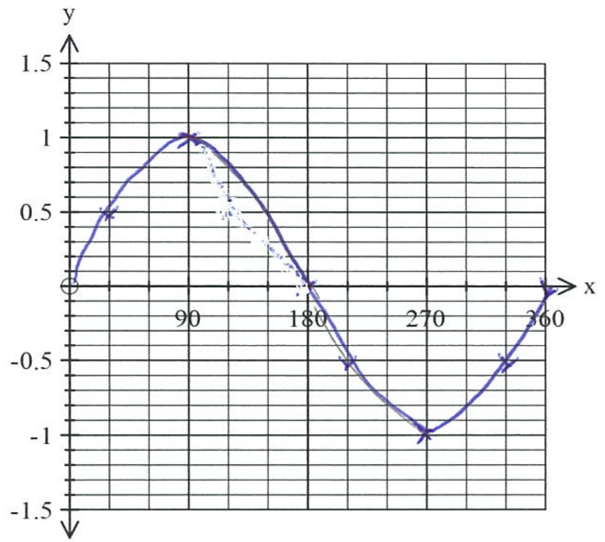
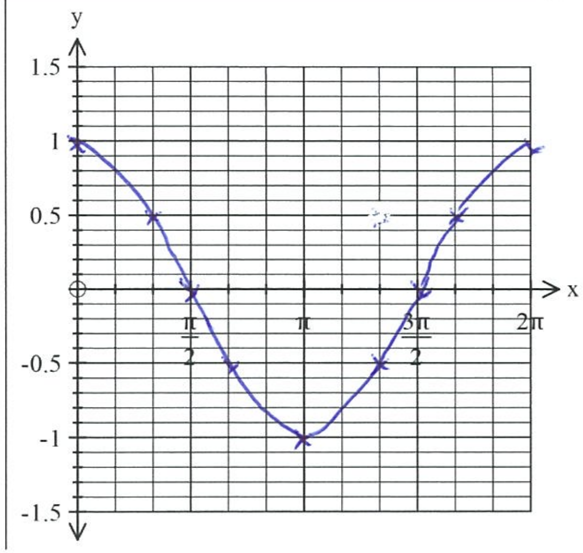
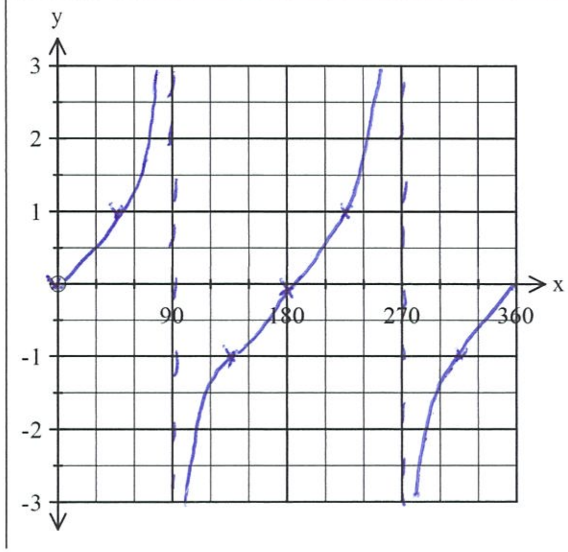
- $y = \sin \boldsymbol{x}$ and $y = \cos \boldsymbol{x}$ have a **period** of 2π; $y = \tan \boldsymbol{x}$ has a period of π
- The **amplitude** of $y = \sin \boldsymbol{x}$ and $y = \cos \boldsymbol{x}$ is 1
- $y = \tan \boldsymbol{x}$ is undefined for the values of $90°$, $270°$ ($\frac{\pi}{2}, \frac{3\pi}{2}$ radians)- this is shown as **asymptotes** on graph
- $y = \sin \boldsymbol{x}$ and $y = \tan \boldsymbol{x}$ are **odd functions** (half turn rotational symmetry around the origin)
- $y = \cos \boldsymbol{x}$ is an **even function** (y axis is a line of symmetry)
- For $y = \sin \boldsymbol{x}$ and $y = \cos \boldsymbol{x}$ the Domain is $x \in$ R ; the Range is $-1 < y < 1$
- For $y = \tan \boldsymbol{x}$ the Domain is $x \in$ R except for multiples of $90°$ or $\frac{\pi}{2}$; the Range is $y \in$ R

Sketching trig graphs

- Graphs can be sketched in degrees or radians. It helps to use the GRAPH function on your graphics calculator.
- Your graphics calculator will automatically be in radians. To change the angle measure, press SHIFT, MENU. Scroll down to Angle and press F1 for DEG. Press EXIT to save.
- To see the entire graph, go SHIFT, F3 (**V-Window**). Change the following settings:

X – min: 0   Y – min: −1.5
max: 420   max: 1.5

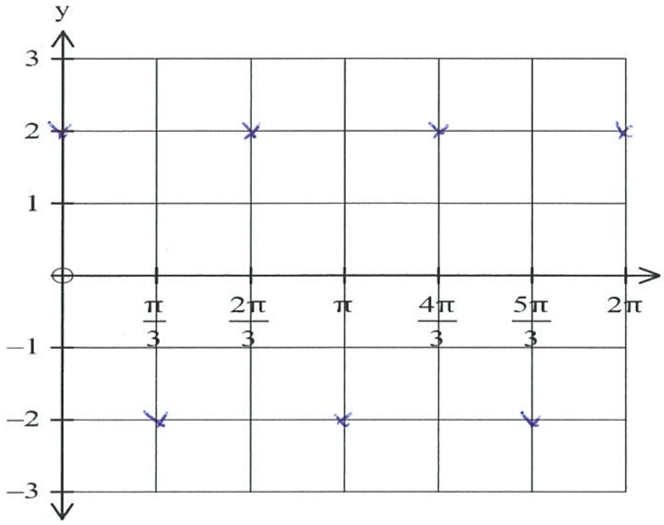
Exercise II: Table of Basic Trigonometric Graphs

Name of graph	$y = \sin x$	$y = \cos x$	$y = \tan x$
Sketch			
***y* – intercept**	(0,0)	(0,1)	(0,0)
***x* – intercepts**	(0,0) (180,0) (360,0)	$(\frac{\pi}{2},0)$ $(\frac{3\pi}{2},0)$	(0,0) (180,0) (360,0)
Amplitude	1	1	–
Period	360°	2π	180°
Special features Odd/even/asymptotes	odd function	even function	asymptotes at 90°, 270°... odd function

Exercise III: Finding Key Points and Sketching Transformed Graphs

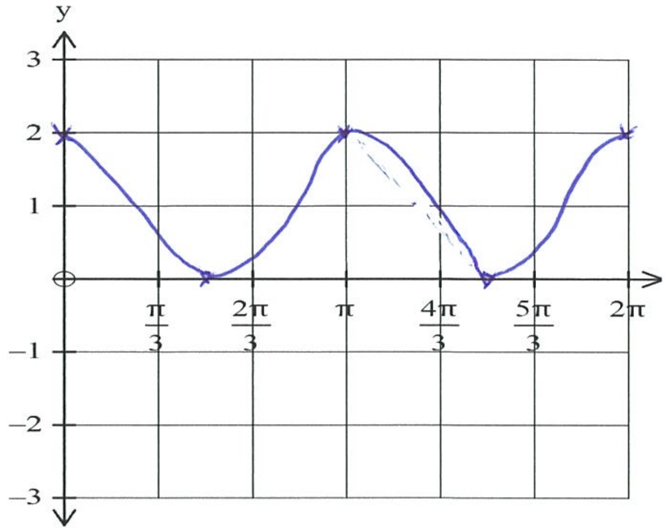
Find the amplitude, period and any horizontal or vertical shift then and then sketch on the grid.

1. $y = 2\cos 3x$



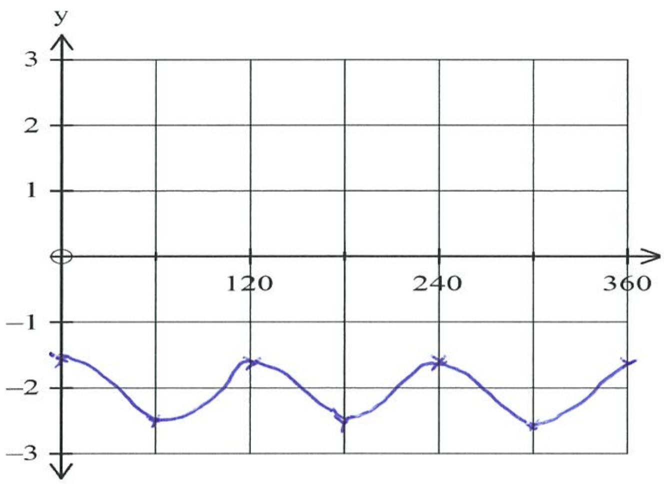
A = 2
period = $\frac{2\pi}{3}$

2. $y = \cos 2x + 1$



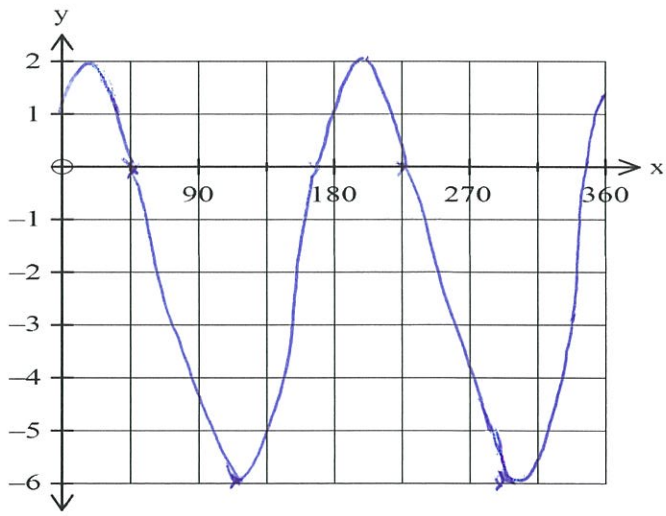
A = 1
period = π
D = +1.

3. $y = \frac{1}{2}\cos 3x - 2$



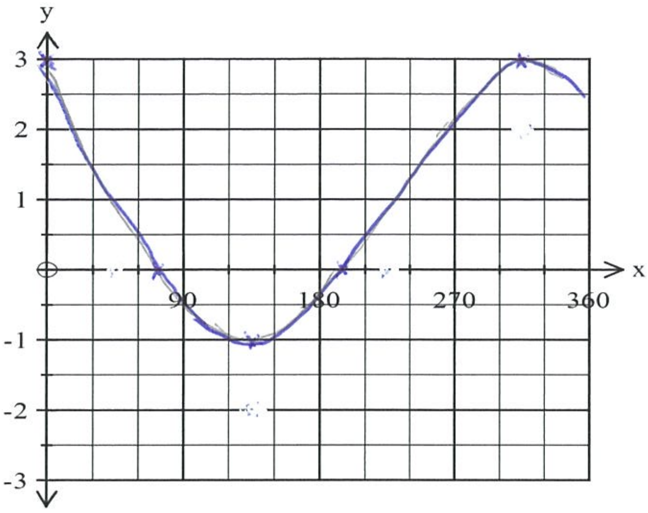
A = $\frac{1}{2}$
period = 120°
D = −2.

4. $y = 4\cos 2(x - 30) - 2$



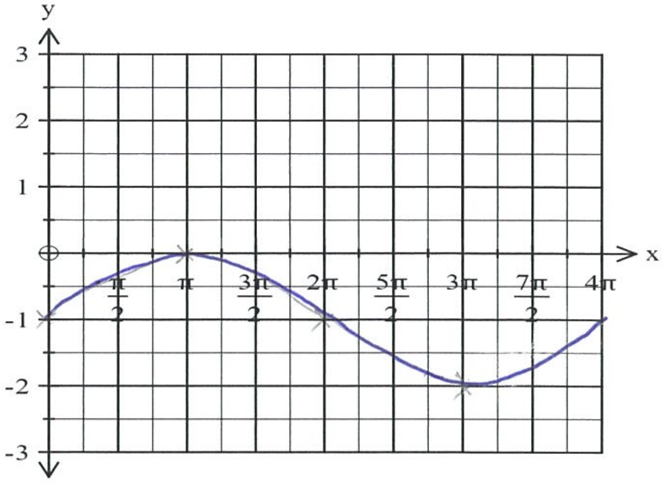
A = 4
period = 180°
horizontal shift 30°
D = −2.

5. $y = 2\cos(x + 45) + 1$



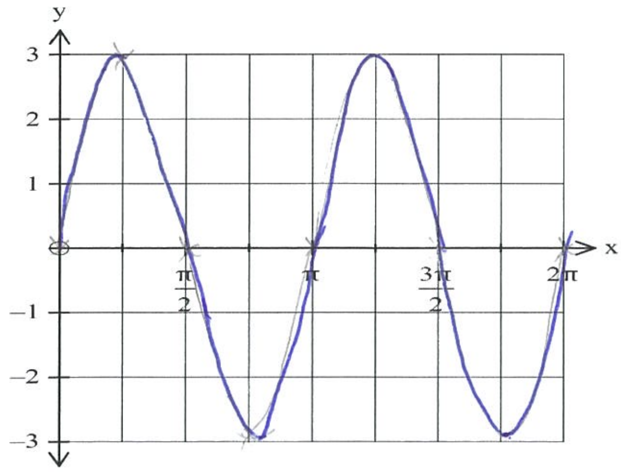
A = 2
period = 360°
D = 1
horizontal shift -45°

6. $y = \sin\frac{1}{2}x - 1$



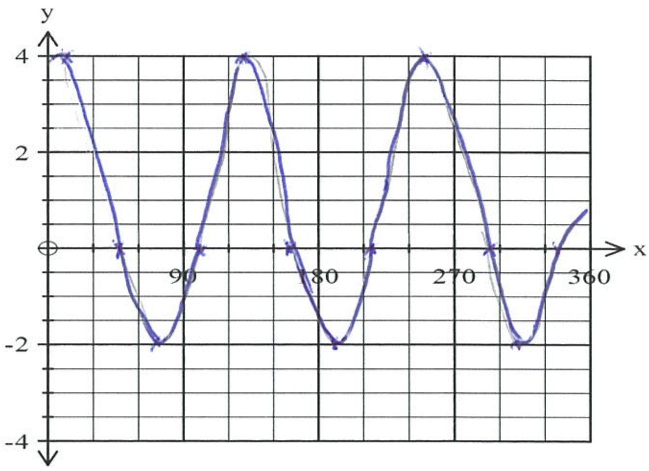
A = 1
period = 4π
D = -1
horizontal : 0.

7. $y = 3\sin 2x$



A = 3
period = 2.

8. $y = 3\sin 3(x + 20) + 1$

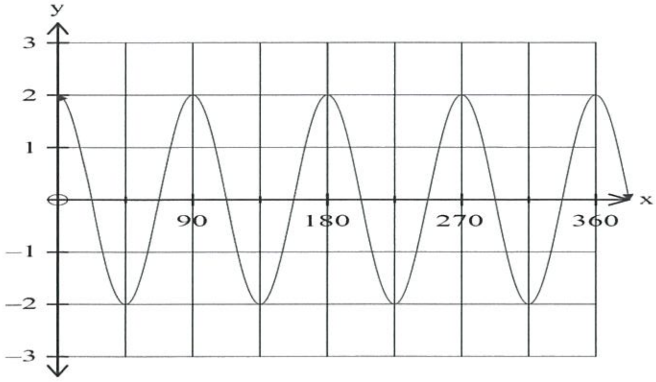


A = 4
period = 120
horizontal -20°
D = 1.

Exercise IV: Writing Equations from Trigonometric Graphs

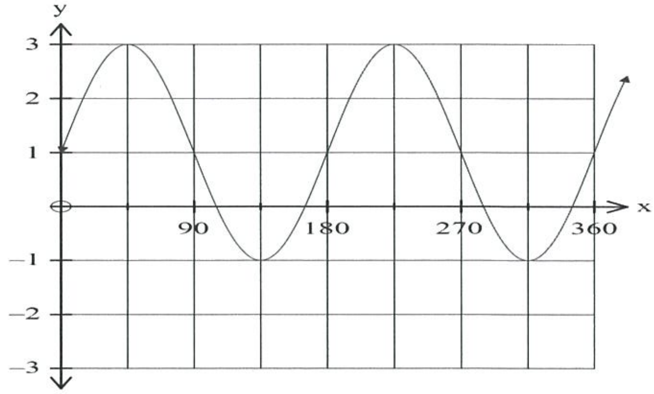
Write trigonometric equations for the following graphs. Check your solution using your graphics calculator.

1



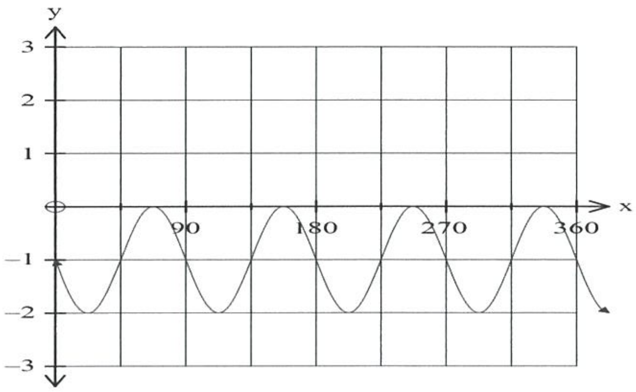
$y = 2\cos 4x$

2



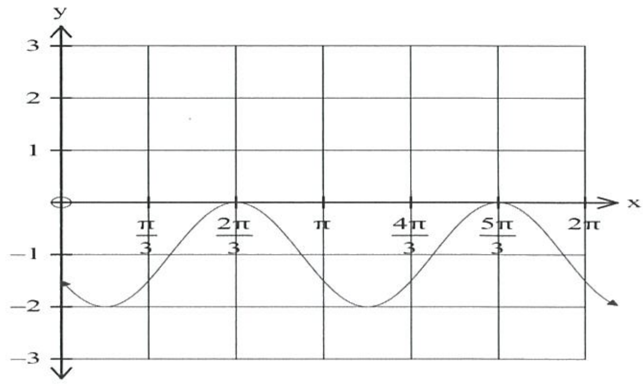
$y = 2\sin 2x + 1$

3



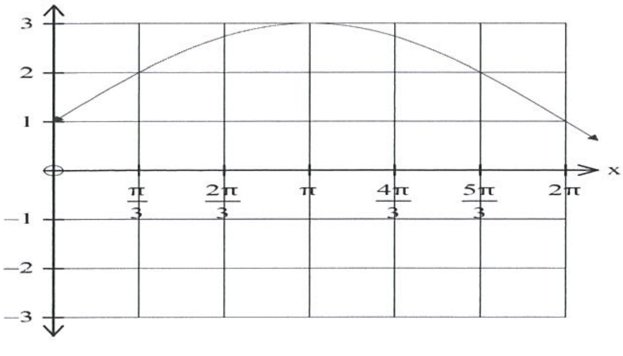
$y = -\sin 4x - 1$

4



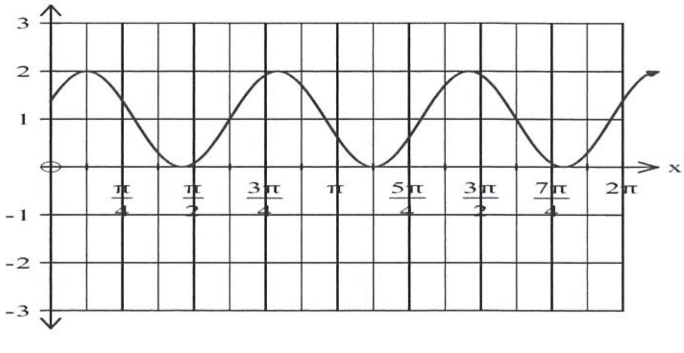
$y = \sin 2\left(x - \frac{5\pi}{2}\right) - 1$

5



$y = 2\sin \frac{1}{2}x + 1$

6



$y = \cos 3\left(x - \frac{\pi}{8}\right) + 1.$

Example: Solve $\cos x + 2 = 1.5,\ 0 \leq x \leq 360^\circ$

Graphics Calculator

The **V-window** should be set to
X – min: 0 X – max: 360°
Y – min: 1 Y – max: 3

There is a vertical shift in the graph

In the graph function, draw:
- $y = \cos x + 2$
- $y = 0.5$

Find the intercepts by pressing SHIFT, F5, F5 (**ISCT**)

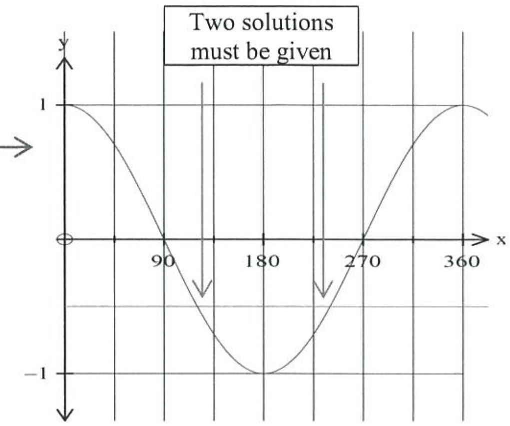
Algebraically

$\cos x + 2 = 1.5$
$\cos x = -0.5$ Draw diagram
$x = \cos^{-1}(-0.5)$
$x = 120^\circ$

Since the cosine graph is symmetrical between 0 and 360°, another solution must exist

$x = 360 - 120 = 240^\circ$

Therefore, $x = 120^\circ$ and 240°



Always draw a diagram when solving trigonometric equations

Exercise V: Solving Trigonometric Equations

Using algebraic methods solve the following trigonometric equations in the specified domain. Space has been provided for you to sketch a diagram of the trigonometric function and the line it intersects with.

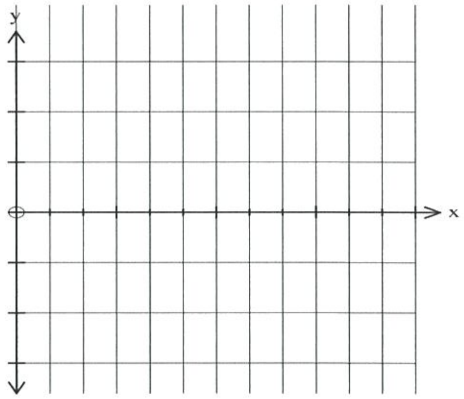
Check your solutions using your graphics calculator.

ONE

$\cos x = 0.3,\ 0 \leq x \leq 2\pi$

$x = \cos^{-1}(0.3)$

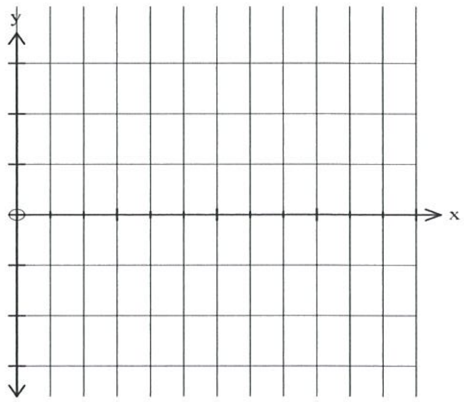
$x = 1.266 \quad 5.017$



TWO

$\cos 2x = 0.3,\ 0 \leq x \leq 2\pi$

$\cos 2x = 0.3$
$2x = 1.266 \quad 5.017\ldots$
$x = 0.633 \quad 2.509$
$3.775,\ 5.65$

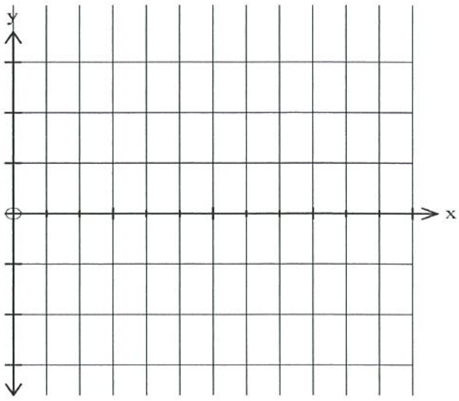


THREE

$\cos(x + 1.2) = 0.3,\ 0 \leq x \leq 2\pi$

$x + 1.2 = 1.266 \quad 5.017$

$x = 0.66 \quad 3.817$



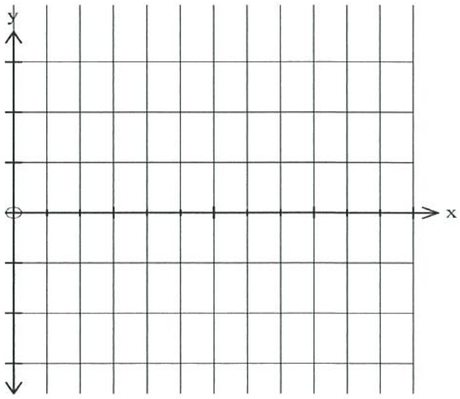
FOUR

$\cos x - 1 = -1.2,\ 0 \leq x \leq 2\pi$

$\cos x = -0.2$

$x = \cos^{-1}(-0.2)$

$x = 1.772 \quad 4.511$

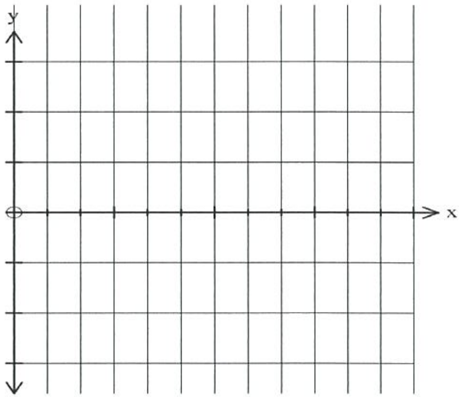


FIVE

$\sin x = -0.45,\ 0 \leq x \leq 360^\circ$

$x = \sin^{-1}(-0.45)$

$x = 206.7,\ 333.3$



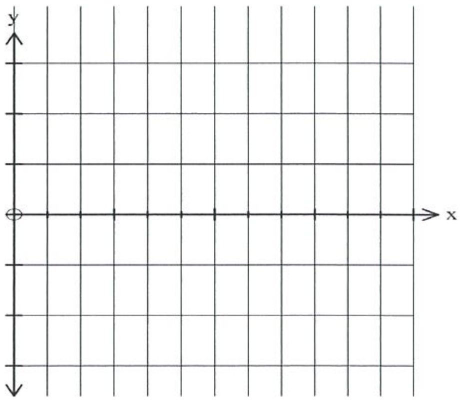
SIX

$\sin 3x = 0.2,\ 0 \leq x \leq 180^\circ$

$3x = \sin^{-1}(0.2)$

$x = 3.85^\circ,\ 56.15,\ 123.85$

$176.15.$

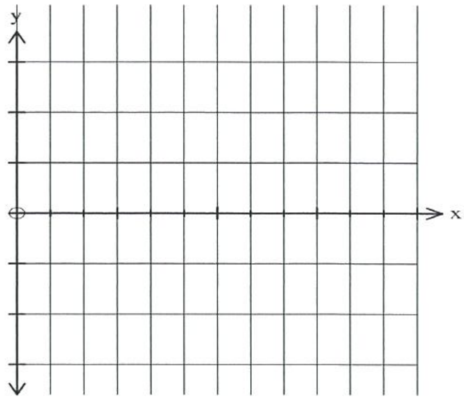


SEVEN

$\sin(x - 2.1) = 0.62,\ 0 \leq x \leq 2\pi$

$x - 21 = \sin^{-1}(0.62)$
$= 0.6687$

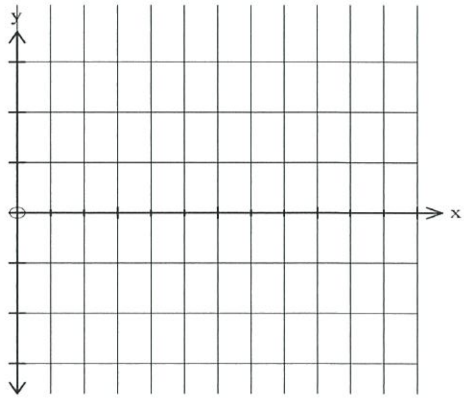
$x = 2.7687 \quad 4.573$



EIGHT

$\sin x + 2 = 1.86,\ 0 \leq x \leq 2\pi$

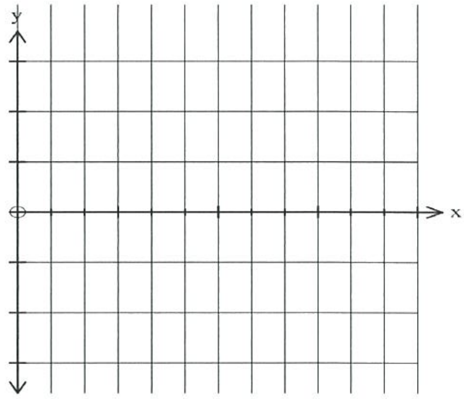
$\sin x = -0.14$
$x = 3.282,\ 6.143.$



NINE

$2 \sin x = -1.8,\ 0 \leq x \leq 360^{\circ}$

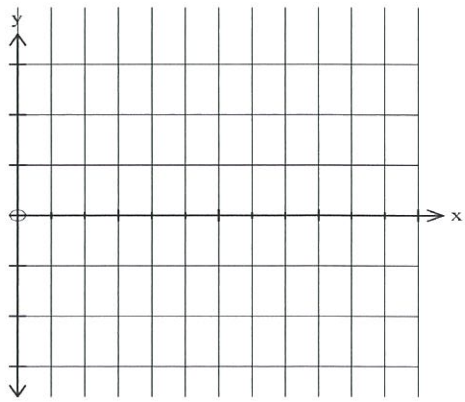
$x = 244.16 \quad 295.84$



TEN

$\cos 4x = 0.3,\ 180^{\circ} \leq x \leq 360^{\circ}$

$x = 198.1^{\circ},\ 251.9^{\circ},\ 288.1^{\circ},$
341.9°

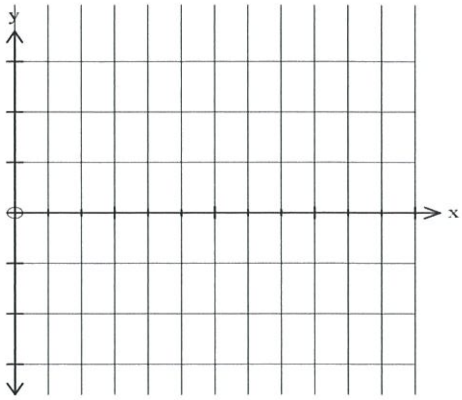


ELEVELN

$3\cos x - 2 = -0.5, -180^\circ \leq x \leq 180^\circ$

$\cos x = 0.5$

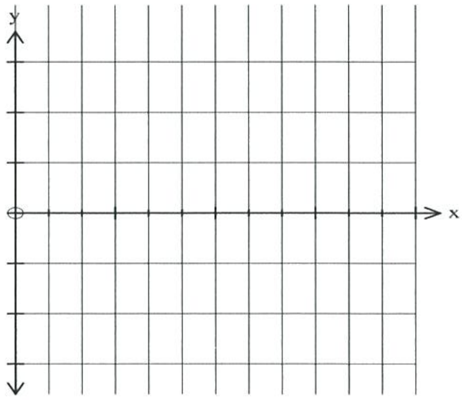
$x = -60^\circ, 60^\circ$



TWELVE

$2\sin\left(x - \frac{\pi}{3}\right) = 1.18,\ 0 \leq x \leq 2\pi$

$x = 1.678 \quad 3.558$



Solving Trigonometric Equations 2

To solve trigonometric equations where there are multiple transformations to the trigonometric function, we want to remove as many transformations as possible, before sketching the diagram and solving.

Example: Solve $3\cos 2(x + 20) + 2 = 4.2,\ 0 \leq x \leq 180^\circ$

<u>Graphics Calculator</u>

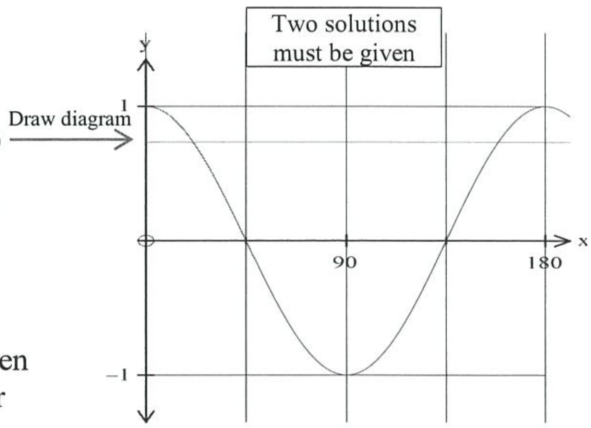
The **V-window** should be set to
X – min: 0 X – max: 180°
Y – min: $2 - 3 = -1$
Y – max: $2 + 3 = 5$

In the graph function, draw:
- $y = 3\cos 2(x + 30) + 2$
- $y = 4.2$

Find the intercepts by pressing SHIFT, F5, F5 (**ISCT**)

<u>Algebraically</u>

$3\cos 2(x + 20) + 2 = 4.2$
$3\cos 2(x + 20) = 2.2$
$\cos 2(x + 20) = 0.7333$
Let $w = x + 20$
$\cos 2w = 0.7333$
$2w = 42.83^\circ$
$w = 21.42^\circ$



Due to the symmetry between 0° and 180°, there is another solution.

$w = 180 - 21.42 = 158.58^\circ$

$x = 21.42 - 20 = 1.42^\circ$ and
$158.58 - 20 = 138.58^\circ$

Always draw a diagram when solving trigonometric equations

Exercise VI: Solving Trigonometric Equations 2

Using algebraic methods solve the following trigonometric equations in the specified domain. Space has been provided for you to sketch a diagram of the trigonometric function and the line it intersects with.

Check your solutions using your graphics calculator.

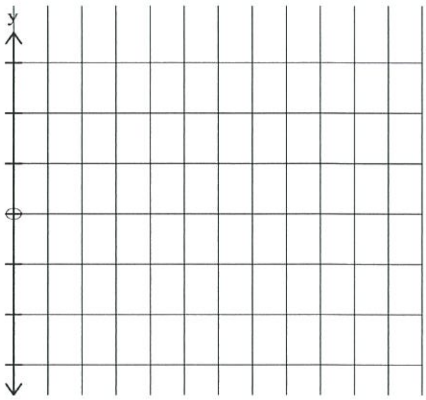
ONE

$5 \sin (x - 15) + 4 = 1.8,\ 0 \leq x \leq 360^{\circ}$

$5\sin(x-15) = -2.2$

$\sin(x-15) = -0.44$

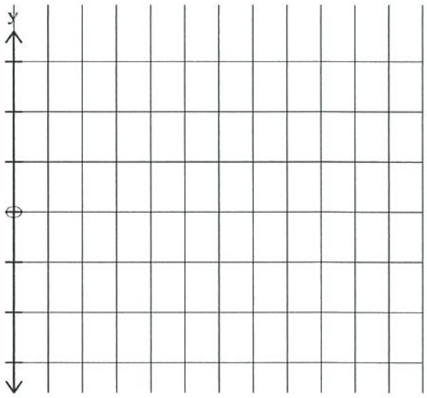
$x = 221.1^{\circ}, 348.9^{\circ}$



TWO

$3\sin 2x - 3 = -1.2,\ 0 \leq x \leq 2\pi$

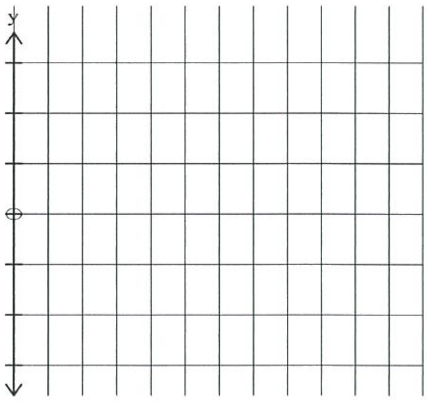
$x = 0.322, 1.249, 3.463$
4.391



THREE

$7 - 2\cos 3\left(x + \frac{\pi}{6}\right) = 6,\ 0 \leq x \leq \pi$

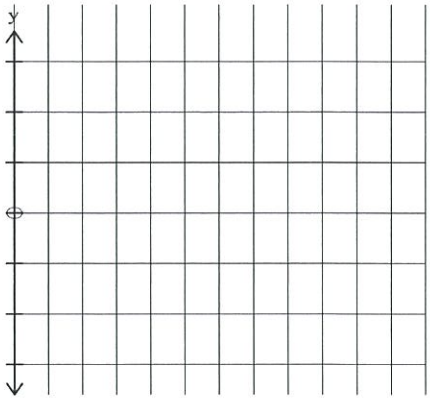
$x = 1.222, 1.920$



FOUR

$12\cos 4\left(x+\frac{\pi}{8}\right)-9=2.8,\ 0\le x\le\pi$

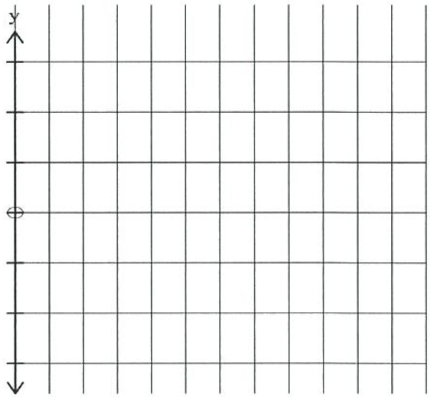
$x = 1.132, 1.224, 2.703$
2.795



FIVE

$15-\sin 2(x-45)=14.2,\ 0\le x\le 180^{\circ}$

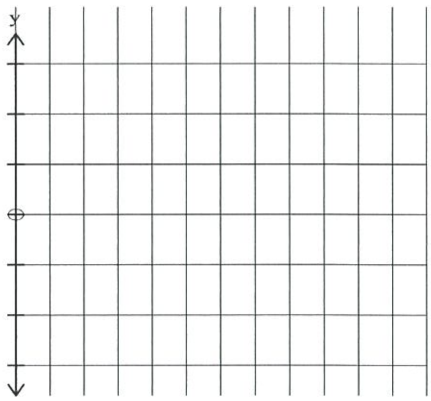
$x = 71.6^{\circ}, 108.4^{\circ}$



SIX

$14\sin\frac{1}{2}(x+15)+3=10,\ 0\le x\le 720^{\circ}$

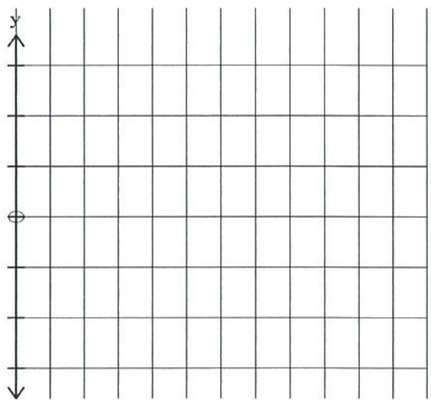
$x = 45^{\circ}$



SEVEN

$5\cos 3(x-1.25)+20=24,\ 0\le x\le 2\pi$

$x = 1.035, 1.465, 3.13, 3.559$
$5.224, 5.653$



Exercise VII: General Solutions

Write the general solution of the following equations and give the x-values for $n = 0, 1, 2$ and 3. Check your solutions on your graphics calculator.

$8 \cos 3x = 6$ in radians

ONE

$\cos 3x = 0.75$

$3x = 2n\pi \pm 0.7227$

$x = \frac{2n\pi}{3} \pm 0.2409$

$x = 0.2409, 1.853, 2.335$

$3.948, 4.48, 6.042$

6.524

$2 \sin x - 2 = -0.5$ in degrees

TWO

$2 \sin x = 1.5$

$\sin x = 0.75$

$x = n\pi + (-1)^n 0.8481$

$x = 0.8481, 2.293, 7.131$

8.577

$\sin 3(x + 180^\circ) + 15 = 15.5$ in degrees

THREE

$\sin 3(x + 180^\circ) = 0.5$

$3(x + 180^\circ) = 180n + (-1)^n . 30$

$x + 180^\circ = 60n + (-1)^n . 10$

$x = 60n + (-1)^n . 10 - 180^\circ$

$x = -170^\circ, -130^\circ, -50^\circ, -10^\circ$

$9\cos\left(x - \frac{4\pi}{5}\right) - 2 = 6.4$ in radians

FOUR

$\cos\left(x - \frac{4\pi}{5}\right) = 8.4$

$\cos\left(x - \frac{4\pi}{5}\right) = \frac{14}{15}$

$x - \frac{4\pi}{5} = 2n\pi \pm 0.3672$

$x = 2n\pi \pm 0.3672 + \frac{4\pi}{5}$

$x = 2.880, 2.146, 8.429, 9.164$

$15.447, 14.712, 20.996, 21.7$

$17\cos 2(x - 62^\circ) + 25 = 10.5$ in degrees

FIVE

$\cos 2(x - 62^\circ) = \frac{-29}{34}$

$2(x - 62^\circ) = 360n \pm 148.5^\circ$

$x - 62^\circ = 180n \pm 74.27^\circ$

$x = 180n \pm 74.27 + 62^\circ$

$x = 136.27, -12.27, 316.27, 167.73$

$496.27, 347.73, 676.27, 527.73$

$45 - 6\sin\frac{\pi}{20}(x + 7.5) = 41.6$ in radians

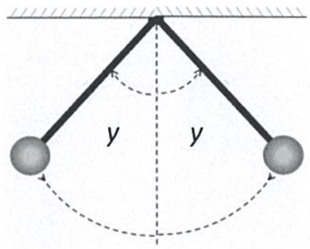
SIX

$x = -3.665, 8.6647, 36.335$

48.665

Exercise VIII: Applications of Trigonometric Graphs

The angle a swinging pendulum in a vacuum makes with a vertical line (in either direction) can be modelled by a trigonometric function.



- The pendulum is released at an angle of 40 degrees. This is the maximum angle.
- The minimum angle possible is zero
- The pendulum completes a swing (from left to right, and back to starting position) 4 times per second.

Use the above information to give the equation of the model, and therefore the angle the pendulum is at 0.2 seconds after the pendulum is released.

ONE

Max = 40° period = 0·25s.
Min = 0

$y = A\cos Bx + D$

$y = 20\cos(8\pi x) + 20$

$D + A = 40$ $A = 20$
$D - A = 0$ $D = 20$
$B = 2\pi \div 0{\cdot}25 = 8\pi$

$x = 0{\cdot}2$
$y = 26{\cdot}2°$

At a certain beach, there is a height marker 1m out from the foreshore. The day Geoff planned to go windsurfing, the water height at this point could be modelled using a trigonometric function.

Geoff starts recording the height of the waves at 8 o'clock in the morning, when the waves are at a maximum of 2.75m. 6 hours later, the waves are at a minimum height of 0.25m.

Geoff does not like launching when the water at this point is over his neck, a height of 1.5m. If the waves maintain this trigonometric model the following day, at what time will Geoff have his launching opportunities during the day, and for how long will this last?

TWO

t = 0 (8am) max 2·75
t = 6 (2pm) min 0·25
period = 12hrs

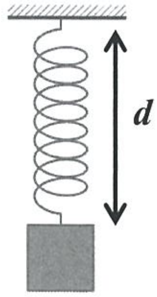
$h = 1{\cdot}25\cos(\frac{\pi}{6}t) + 1{\cdot}5$
$h < 1{\cdot}5$
t = 3 - 9 hours after 8am
15 - 21 " " " ← too late in day

$A = 1{\cdot}25$ $D = 1{\cdot}5$
$B = 2\pi \div 12$
$= \frac{\pi}{6}$.

earliest launch 11am til 5pm

THREE

A block is attached to a spring. The spring is extended and released.



The distance, ***d*** centimeters, of the top of the block below the spring's attachment point ***t*** seconds after release can be modelled by a trigonometric equation

It takes the block 4 seconds to return to its starting point.
The closest it gets to its attachment point is 2.5 cm and the furthest is 8.5 cm.
Find the equation for *d* and use it to find when the block is first 4 cm from the attachment point.

$t=0$ max $= 8.5$ cm
min $= 2.5$ cm
period 4s.

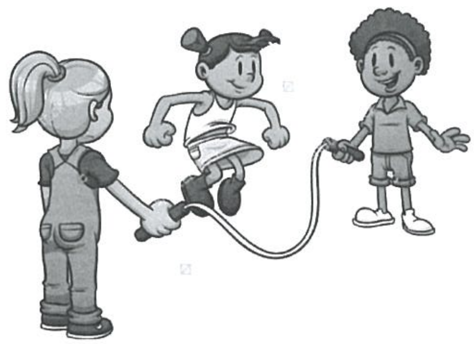
$d = 3\cos\left(\frac{\pi t}{2}\right) + 5.5$

$d=4$ $t = 1.33$ s - first time

$A=3$, $D=5.5$ $B=\frac{\pi}{2}$.

FOUR

Two people are turning a skipping rope. The height of the rope handle (h) above the ground at t seconds after the rope starts to turn is modelled by a trigonometric equation. At the lowest point the handle is 66cm above the ground. It reaches a maximum height of 190c m above the ground. One complete turn takes 1.8 seconds.



When would the rope by 1.4m, or higher above the ground?

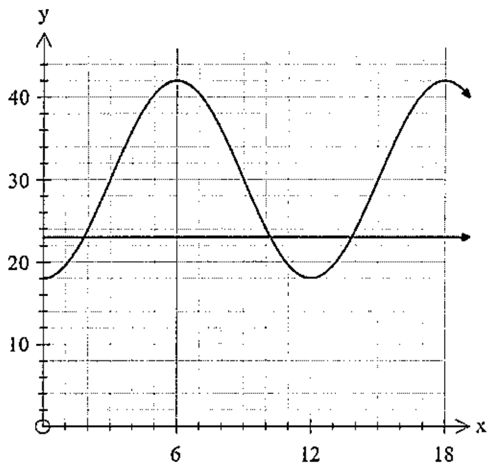
$t=0$ min $= 66$cm
$t=0.9$ max $= 190$cm.
period $= 1.8$ sec.

$h = -62\cos\left(\frac{10\pi}{9}\right) + 128$

$h=140$ (above)
t: $0.51 < t < 1.29$ sec.

$D+A = 190$
$D-A = 66$
$A = 62$, $D=128$ $B = \frac{2\pi}{1.8}$
$= \frac{10\pi}{9}$.

Kittens – solution



Toy 1

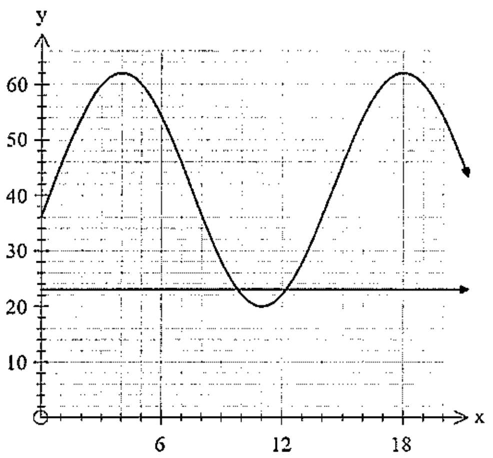
A = 12 $\quad$ B = $\frac{2\pi}{12} = \frac{\pi}{6}$

C = -6 $\quad$ D = 30

$y = 12\cos(\frac{\pi}{6}(t - 6)) + 30$

OR

$y = 12\sin(\frac{\pi}{6}(t - 3)) + 30$



Toy 2

A = 21 $\quad$ B = $\frac{2\pi}{14} = \frac{\pi}{7}$

C = -3 $\quad$ D = 30

$y = 21\cos(\frac{\pi}{7}(t - 4)) + 41$

Toy 1 (feather):

$12\cos(\frac{\pi}{6}(t - 6)) + 30 = 23$

$\cos(\frac{\pi}{6}(t - 6)) = \frac{-7}{12}$ $\qquad$ t = 1.8, 10.2, 13.8……

Solution $\quad$ $0 < t < 1.8$; $\quad$ $10.2 < t < 13.8$ (this part can be done by GC)

General solution – Toy 1

$\cos(\frac{\pi}{6}(t - 6)) = \frac{-7}{12} = -0.5833$ $\alpha = \cos^{-1}(-0.5833) = 2.1936$ $\frac{\pi}{6}(t - 6) = 2n\pi \pm 2.1936$ $t - 6 = 12n \pm 4.189$ $t = 12n + 6 \pm 4.189$	Check n = 0 t = 1.811, 10.189 n = 1 t = 13.811, 22.189 n = 2 t = 25.811, 34.189 n = 3 t = 37.811, 46.189 n = 4 t = 49.811, 58.189 n = 5 t = 61.811, 70.189 n = 6 t = 73.811, 82.189	Liath can reach the feathered toy start – 1.8 sec 10.2 – 13.8 sec 22.2 – 25.81 sec 34.2 – 37.811 sec 46.2 – 49.811 sec He can reach for 3.6 secs then not for 8.4 sec in a repeating pattern

Toy 2 (ball):

$21\cos\left(\frac{\pi}{7}(t-4)\right) + 41 = 23$

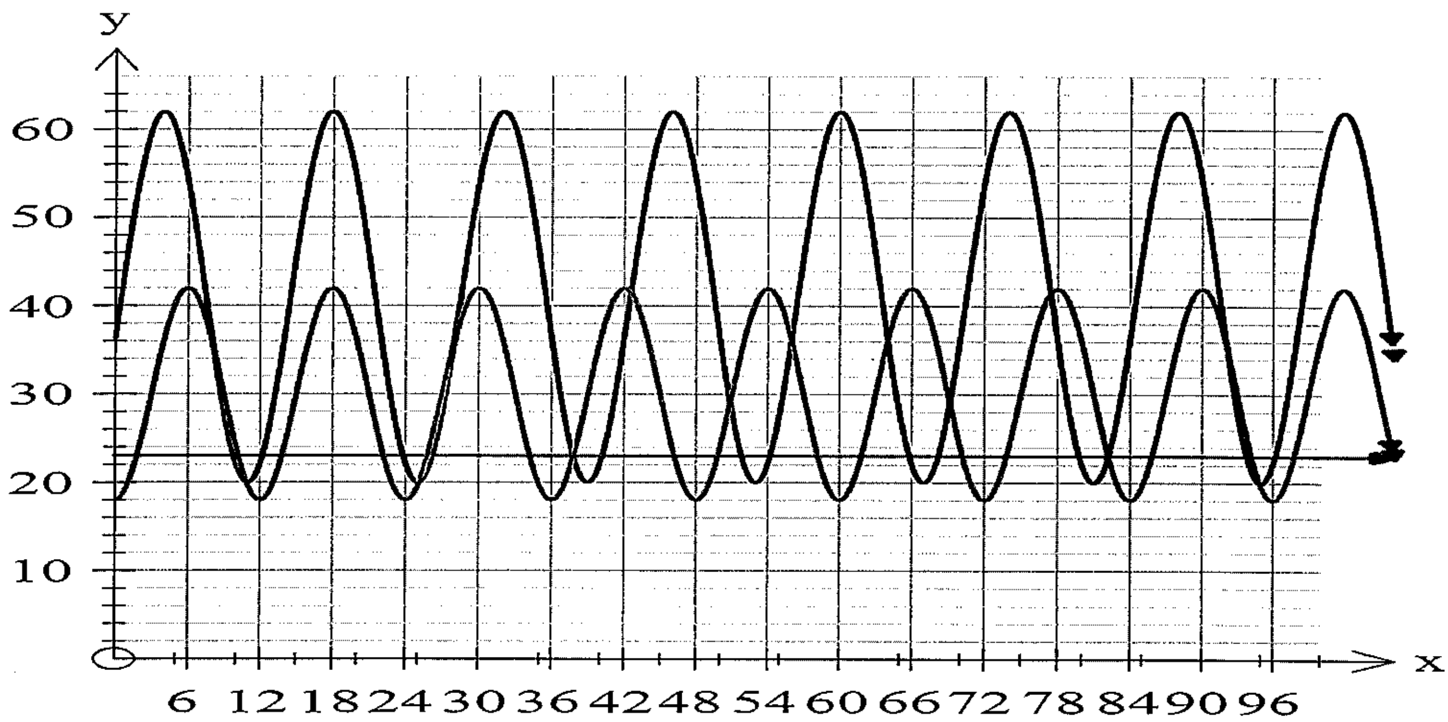
$\cos\left(\frac{\pi}{7}(t-4)\right) = \frac{-18}{21}$ $\quad$ t = 9.8, 12.2, 13.8, 26.2……

Solution $\quad 9.8 < t < 12.21; \quad 23.79 < t < 26.21$

General solution – Toy 2

$\cos(\frac{\pi}{7}(t-4)) = \frac{-18}{21} = -0.8571$ $\alpha = \cos^{-1}(-0.8571) = 2.600$ $\frac{\pi}{7}(t-4) = 2n\pi \pm 2.600$ $t - 4 = 14n \pm 5.794$ $t = 14n + 4 \pm 5.794$	Check n = 0 t = -1.794, 9.794 n = 1 t = 12.21, 23.794 n = 2 t = 26.21, 37.794 n = 3 t = 40.21, 51.794 n = 4 t = 54.21, 65.794 n = 5 t = 68.21, 79.794 n = 6 t = 82.21, 91.794	Liath can reach the ball 9.79 – 12.21 sec 23.79 – 26.21 sec 22.2 – 25.81 sec 37.79 – 40.21 sec 51.79 – 54.21 sec After 9.8 sec he can reach the ball for 2.4 sec then not for 11.6 sec in a repeating pattern

When can Liath reach both toys at the same time?



10.189 – 12.21 $\quad$ The pattern will repeat every 84 seconds (LCM for 12 and 14)

23.79 – 25.811

37.79 – 37.811

40.21 – 46.189

54.21 – 58.189

68.21 – 70.189

73.811 – 79.794